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A Comparative Study of Lagrangian Dynamics and Kinetic Energy in Terms of Work

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ABSTRACT

The ambiguous definition of the Lagrangian and kinetic energy make the concept and notion of the principle of least action misleading and confusing. In this work the Lagrangian and kinetic energy are defined in terms of work. This gives the principle of least action its real physical meaning as reflecting the tendency of physical systems to select a trajectory which consumes less work. The generalized definition of kinetic energy in terms of the work is compatible with that of Newton, Einstein and generalized special relativity. It is very striking to note that the compatibility of this definition with the notion of force requires introducing the mass as an extra dimension extending our 4 space-time coordinates to be having 5 coordinates

Keywords: *Lagrargian, kinetic energy, work, least action, special relativity, Newton's laws.*

I. INTRODUCTION

Theoretical physics aims to describe the physical world that at once concise and comprehensive .The action principle is the one of the widely used formalism. It is based on the fact that the physical system moves from point to another one by selecting the trajectory which make it consume less energy. This action is based on Lagrangian formalism. Lagrangian methods have been extensively studied for the description of physical systems by using the concept and notion of energy [1,2].

Euler-Lagrange equation represent in a general set of differential equation that describe the time evolution of a mechanical system by satisfying the principle of virtual work[3].The Lagrangian formulation of mechanics is an alternative to the classical formalism, which is based on Newton's laws, but leads to the same equations of motion more quickly. Newton's laws are based on the concept of force. If the system is physically constrained, the constraint forces come explicitly into the equations. The Lagrangian formalism is based on energies rather than forces. Also this formalism is independent of coordinate transformations[4].

Lagrange equation can also be derived by using the principle of least action. This principle is based on the fact that any physical system that moves in space, between two points, selects a trajectory that makes it librates minimum energy or absorbs maximum energy from the surrounding. The principle of least action is the back bane of classical mechanics which was reformulated by some of the giants of mathematical physics-people like Lagrange, Euler and Hamilton. This new way of doing things is better for a number of reasons. Firstly, it's elegant. In fact, it's not just elegant it's completely gorgeous and more powerful. It gives new methods to solve hard problems in a fairly straightforward manner. Moreover, it is the best way of exploiting the power of symmetries. It is known that all of physics is based on fundamental symmetry principles. Finally and most importantly, it is universal. It provides a framework that can be extended to all other laws of physics, and reveals a deep relationship between classical mechanics and quantum mechanics. This is the real reason why it's so important. It's the key idea that leads to this new way of thinking. Today, one uses the Lagrangian method to describe all of physics, not just mechanics. All fundamental laws of physics can be expressed in terms of a least action principle. This is true for electromagnetism, special and general relativity, particle physics, and even more speculative pursuits that go beyond known laws of physics such as string theory[5,6,7,8].

II. SIMPLE DERIVATION OF LAGRARGIAN EQUATION

The Lagrargian equation can be simply derived by using Newton's laws. One can use the definition of force in terms of momentum, mv and potential energy V to get:

$$\frac{d}{dt}(mv) = F = -\frac{\partial V}{\partial x} \qquad (1)$$

where the kinetic energy is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \qquad (2)$$

which satisfies

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} = mv \qquad (3)$$

Inserting (3)*in*(1)*gives*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -\frac{\partial V}{\partial x} \qquad (4)$$

Thus rearranging (4)*gives*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) + \frac{\partial V}{\partial x} = 0 \qquad (5)$$

Now define the Lagrangian L to be

$$L = T - V \qquad (6)$$

with the definition of generalized coordinate to be
$q = x$ *one gets*

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{x}} \qquad (7)$$

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial x} = \frac{\partial T}{\partial x} - \frac{\partial V}{\partial x} = -\frac{\partial V}{\partial x} \qquad (8)$$

where Tand V are assumed to be dependent on $\dot{q}$ only, and q only respectively. Thus inserting (7) and (8), one gets the ordinary Lagrangian equation in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad (9)$$

III. THE PHYSICAL MEANING OF L

Unfortunately the formal definition of L in equation (6) is misleading. To remove this ambiguity let us redefine T and V in terms of work W to be

$$T = \int F.dr = W \qquad (10)$$

$$V = -\int F.dr = -W \qquad (11)$$

According to equation (6)
$L = T - V$
Using equations (10) and (11) yields

$$L = 2W \qquad (12)$$

A direct substitution of (12) in (9) gives

$$2\left[\frac{d}{dt}\frac{\partial W}{\partial \dot{q}} - \frac{\partial W}{\partial q}\right] = 0$$

$$\frac{d}{dt}\left[\frac{\partial W}{\partial \dot{q}}\right] - \frac{\partial W}{\partial q} = 0 \qquad (13)$$

Thus the Lagrange function is replaced by the work done. According to equation (12) Lagrange function is double the work done by the system. According to the definition of T in (10), the kinetic energy in special relativity (SR) is given by

$$T = \int F.dr = \int vdmv = mc^2 - m_\circ c^2 = m_\circ c^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - m_\circ c^2 \quad (14)$$

By using the identity, for small x

$$(1+x)^m = 1 + mx \quad (15)$$

Thus using equations (14) and (15) yields

$$T = m_\circ c^2 \left[1 + \frac{1}{2}\frac{v^2}{c^2}\right] - m_\circ c^2$$

$$T = \frac{1}{2} m_\circ v^2 \quad (16)$$

This is the ordinary definition of kinetic energy in Newtonian mechanics. Thus the definition of T in terms of w is more general.

Thus the expression of T which is compatible with SR and Newton's one is given by

$$T = m_\circ c^2 \left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right] \quad (17)$$

Let us now see how T looks like in generalized special relativity (G S R), where

$$T = \int_0^v F.dr = \int_0^v \frac{d(mv)}{dt} dr = \int_0^v d(mv)\frac{dr}{dt}$$

$$= \int_0^v d(mv^2) - \int_0^v mvdv$$

$$= mv^2 - \int_0^v \frac{m_\circ v}{\sqrt{g_{\circ\circ} - \frac{v^2}{c^2}}} dv = mv^2 - I \quad (18)$$

Where

$$m = \frac{m_\circ}{\sqrt{g_{\circ\circ} - \frac{v^2}{c^2}}} \quad (19)$$

To perform integration I, one can define

$$cos\theta^2 = \frac{v^2}{g_{\circ\circ}c^2} \quad , \qquad sin\theta^2 = 1 - cos\theta^2 \quad (20)$$

Thus

$$sin\theta^2 = 1 - \frac{v^2}{g_{\circ\circ}c^2} \quad \Longrightarrow v = c\sqrt{g_{\circ\circ}}.cos\theta$$

$$\frac{dv}{d\theta} = -c\sqrt{g_{\circ\circ}}.sin\theta \quad (21)$$

Hence

$$I = \int_0^v \frac{m_\circ v}{\sqrt{g_{\circ\circ} - \frac{v^2}{c^2}}} dv = -m_\circ c^2 g_{\circ\circ} \int_0^v \frac{cos\theta sin\theta d\theta}{\sqrt{g_{\circ\circ}}\sqrt{1 - \frac{v^2}{g_{\circ\circ}c^2}}}$$

$$= -m_\circ c^2 \sqrt{g_{\circ\circ}} \int_0^v \frac{cos\theta sin\theta d\theta}{\sqrt{1 - \frac{v^2}{g_{\circ\circ}c^2}}} \quad (22)$$

Therefore

$$= -m_\circ c^2 \sqrt{g_{\circ\circ}} \int_0^v \frac{cos\theta sin\theta d\theta}{\sqrt{sin\theta^2}} = -m_\circ c^2 \sqrt{g_{\circ\circ}} \int_0^v \frac{cos\theta sin\theta d\theta}{sin\theta}$$

$$= -m_\circ c^2 \sqrt{g_{\circ\circ}} \int_0^v \cos\theta d\theta \quad (23)$$

But

$$\frac{dsin\theta}{d\theta} = cos\theta \quad \Longrightarrow dsin\theta = cos\theta d\theta \qquad (24)$$

Thus

$$I = -m_\circ c^2 \sqrt{g_{\circ\circ}} \int_0^v cos\,\theta d\theta = -m_\circ c^2 \sqrt{g_{\circ\circ}} \int_0^v dsin\theta \qquad (25)$$

$$= -[m_\circ c^2 sin\theta]_0^v \sqrt{g_{\circ\circ}} = -m_\circ c^2 \sqrt{g_{\circ\circ}} \left[\left[1 - \frac{v^2}{g_{\circ\circ} c^2}\right]^{\frac{1}{2}}\right]_0^v$$

$$= -m_\circ c^2 \sqrt{g_{\circ\circ}} \left[\left[\frac{1}{g_{\circ\circ}}\left(g_{\circ\circ} - \frac{v^2}{c^2}\right)\right]^{\frac{1}{2}}\right]_0^v = -m_\circ c^2 \left[\left[g_{\circ\circ} - \frac{v^2}{c^2}\right]^{\frac{1}{2}}\right]_0^v$$

$$I = -m_\circ c^2 \left(g_{\circ\circ} - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m_\circ c^2 \sqrt{g_{\circ\circ}} = \frac{-m_\circ c^2}{\sqrt{g_{\circ\circ} - \frac{v^2}{c^2}}}\left(g_{\circ\circ} - \frac{v^2}{c^2}\right) + m_\circ c^2 \sqrt{g_{\circ\circ}}$$

$$I = -mc^2 \left(g_{\circ\circ} - \frac{v^2}{c^2}\right) + \sqrt{g_{\circ\circ}}\, m_\circ c^2 = -mc^2 g_{\circ\circ} + mv^2 + \sqrt{g_{\circ\circ}}\; m_\circ c^2 \qquad (26)$$

Inserting equation (26) in (18) yields

$$T = \int_0^v d(mv)\frac{dr}{dt} = \int_0^v d(mv^2) - \int_0^v mvdv$$

$$= mv^2 - \int_0^v mv\,dv \qquad (27)$$

Thus the kinetic energy is given by

$$T = mv^2 - (-mc^2 g_{\circ\circ} + mv^2) + \sqrt{g_{\circ\circ}}\, m_\circ c^2 = g_{\circ\circ} mc^2 - \sqrt{g_{\circ\circ}}\, m_\circ c^2 \qquad (28)$$

Since

$$g_{\circ\circ} = \left(1 + \frac{2\emptyset}{c^2}\right) \qquad (29)$$

Thus the kinetic energy consists of a potential term which does not agree with the formal definition of T when one consider SR limit $g_{\circ\circ} = 1$ thus equation (29) become

$$T = mc^2 - m_\circ c^2 \qquad (30)$$

However for: $\frac{\emptyset}{c^2}, \frac{v^2}{c^2} \ll 1$ equation (28) gives

$$T = \left(1 + \frac{2\emptyset}{c^2}\right)\left(1 + \frac{2\emptyset}{c^2} - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} m_\circ c^2 - \left(1 + \frac{2\emptyset}{c^2}\right)^{\frac{1}{2}} m_\circ c^2$$

$$T = m_\circ c^2 - m_\circ \emptyset + \frac{1}{2} m_\circ v^2 + 2m_\circ \emptyset - m_\circ c^2 - m_\circ \emptyset = \frac{1}{2} m_\circ v^2 \qquad (31)$$

The dependence of T on the generalized coordinates can be found by using the definition of force in (1) and (4) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}(mv) = F \qquad (32)$$

Thus:

$$\frac{\partial T}{\partial v} = mv \qquad (33)$$

But according to the general definition of T in equation (10)

$$T = \int \underline{F}.d\underline{r} = \int \frac{d(mv)}{dt}.d\underline{r} = \int vd(mv)$$

$$T = \int mvdv + \int v^2 dm \qquad (34)$$

If one assumes T to depend on velocity v and mass m, i.e.

$$T = T(v, m) = T(\dot{q}, m) \qquad (35)$$

It follows that

$$dT = \frac{\partial T}{\partial v} dv + \frac{\partial T}{\partial m} dm \qquad (36)$$

Hence

$$T = \int dT = \int \frac{\partial T}{\partial v} dv + \int \frac{\partial T}{\partial m} dm \qquad (37)$$

Comparing equations (34) and (37) yields

$$\frac{\partial T}{\partial v} = mv \qquad \frac{\partial T}{\partial m} = v^2 \qquad (38)$$

Thus the mass enter as a generalized coordinate.

IV. DISCUSSION

The ordinary definition of the Lagrangian L in terms of kinetic energy T and the potential energy V is ambiguous and confusing, where

$$L = T - V$$

This definition is really misleading and confusing. This needs new view of the real physical meaning of the Lagrangian. To do this, one able redrives Lagrangian equation using simple argument. According to the definition of a force F in terms of momentum, kinetic energy T and potential energy V as shown by equations$(1, -, 9)$.

In view of equations (10), (11) and (12), the Lagrangian L is given by

$$L = 2w$$

This means that the Lagrangian physically means the work done by the system. But according to definition (12) the work done can be redefined to be

$$W = \frac{1}{2} L = \frac{1}{2}(T - V)$$

But since equations (7) and (8) shows that

$$T = T(\dot{q})$$

$$V = V(q)$$

Therefore

$$W = W(x, q, \dot{q})$$

However definition (34) gives generalized coordinate extra mass dimension, which forces us to write

$$W = W(x, q, \dot{q}, m)$$

It is very interesting to note that Lagrange equation (9) can be replaced by work equation which gives a direct physical meaning to Lagrangian and principle of least action. Where this principle now means that the physical system chooses the path that enables it to do minimum energy. The definition of kinetic energy T in terms of the work done W in equation (14) make T conforms to SR and Newtonian definition as shown by equations (14) and (16). However for GSR the definition is confusing since T consists of potential term as equation (18) reads. But strikingly the expression of T in GSR reduces to the ordinary SR and Newtonian expression for T as shown by equations (30) and (31).The new definition of T in terms of W conforms to the definition of force in Newtonian laws (equations (32… 38)) only when the mass is introduced as an extra generalized coordinate. This means that mass may represent the fifth dimension which enables understanding the physics better.

V. CONCLUSION

The Lagrargian is shown to be reprinting the work done by the system. This gives the principle of least action its real physical meaning. Where it means the tendency of the system to select a trajectory that allows minimum work. The definition of T in terms of W conforms to SR and Newtonian and forces include mass as an extra dimension.

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